

# Multiple solutions for a class of Kirchhoff equation with singular nonlinearity

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**Abstract.** In this article, we investigate the existence and multiplicity of solutions of Kirchhoff equation

$$\begin{cases} -(1+b \int_{\mathbb{R}^3} |\nabla u|^2) \Delta u = k(x) \frac{|u|^2 u}{|x|} + \lambda h(x) u, & x \in \mathbb{R}^3 \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty \end{cases}$$

where the potential  $k(x)$  allows sign changing. Making use of Nehari manifold method and concentration-compactness principle, we obtain the existence and multiplicity of solutions for this equation. Our main results can be viewed as partial extensions of the results of [11, 12, 17].

*Keywords:* Kirchhoff equation; Indefinite weight ; concentration-compactness principle; Nehari manifold; Singular nonlinearity.

## 1 Introduction

The system

$$\begin{cases} -(a+b \int_{\mathbb{R}^3} |\nabla u|^2) \Delta u + V(x)u = f(x, u) & x \in \mathbb{R}^3 \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty \end{cases} \quad (1.1)$$

is related to the stationary analogue of the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{p_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial t} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \quad (1.2)$$

which was presented by Kirchhoff in 1883. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. The parameters in (1.2) have following physical meanings:  $L$  is the length of the string,  $h$  is the area of the cross-section,  $E$  is the Young modulus of the material,  $\rho$  is the mass density and  $P_0$  is the initial tension. After J. L. Lions's work [16], which introduced an abstract functional analysis framework to the following equation

$$u_{tt} - (a+b \int_{\Omega} |\nabla u|^2) \Delta u = f(x, u), \quad (1.3)$$

equation (1.1) received much attention. See [2, 4, 19] and the references therein. A typical way to deal with equation (1.1) is to use the mountain pass theorem [3]. For this purpose, one usually assumes that

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$f(x, u)$  is subcritical, superlinear at the origin and satisfies the Ambrosetti-Rabinowitz condition (AR in short): e.g. see [13]

$$\exists \mu > 4 \text{ such that } 0 < \mu F(x, u) \leq f(x, u)u \text{ for all } u \in \mathbb{R}.$$

Using Nehari manifold method, He and Zou [14] proved the existence of positive ground state solution of (1.1) with the nonlinearity satisfying the Ambrosetti-Rabinowitz condition. The typical case is  $f(u) \sim |u|^{p-2}u$  with  $4 < p < 6$ . Wu [20] obtained the existence of nontrivial solutions to a class of Kirchhoff equation. He assumed that the nonlinearity  $f(x, u)$  is 4-superlinear at infinity and satisfies

$$4F(x, u) \leq f(x, u) \text{ for all } u \in \mathbb{R}.$$

In order to get compactness, he considered the problem in a weight subspace

$$E \triangleq \{u \in H^1 \mid \int_{\mathbb{R}^3} V(x)|u|^2 dx < \infty\}$$

such that  $E \hookrightarrow L^p$  is compact. Li and Ye [15] partially extended the results of He and Zou to  $3 < p < 6$  by monotonicity trick and a global compactness lemma. There are also some works to deal with Kirchhoff equation with indefinite nonlinearity. Recently, Chen, Kuo and Wu [11] investigated the multiplicity of positive solutions for the problem which involving sign-changing weight functions

$$\begin{cases} -(a + b \int_{\mathbb{R}^3} |\nabla u|^2) \Delta u = k(x)|u|^{p-2}u + \lambda h(x)|u|^{q-2}u, & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases} \quad (1.4)$$

where  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^3$  with  $1 < q < 2 < p < 6$ . The authors showed that existence and multiplicity of results strongly depend on the size of  $p$  with respect to 4. Part of the results is the following: If  $p = 4$ , then the problem has (at least) one solution for  $b$  large and two positive solutions for  $b$  and  $\lambda$  small. Chen [12] proved that equation

$$\begin{cases} -(1 + b \int_{\mathbb{R}^3} |\nabla u|^2) \Delta u = k(x)|u|^{p-2}u + \lambda h(x)u, & x \in \mathbb{R}^3 \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty \end{cases} \quad (1.5)$$

exists multiple positive solutions, where  $k(x)$  allows sign changing with  $p \in (4, 6)$ . As for singular nonlinearity, Liu and Sun [17] considered the existence of positive solutions for the following problem with singular and superlinear terms

$$\begin{cases} -(a + b \int_{\mathbb{R}^3} |\nabla u|^2) \Delta u = h(x)u^{-r} + \lambda k(x) \frac{|u|^{p-2}u}{|x|^s} & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases} \quad (1.6)$$

where  $0 \leq s < 1$ ,  $4 < p < 6 - 2s$ ,  $0 < r < 1$  and  $k(x) \geq 0$ . They obtained two positive solutions by Nehari manifold. However, very little is known for existence of nontrivial solutions of (1.1) if  $f(x, u)$  is singular and indefinite. Motivated by [17, 11, 12], in the present paper, we consider the case where  $f(u, x)$  is a combination of a singular 4-linear term and a linear term. More precisely, we study the following system with the form

$$\begin{cases} -(1 + b \int_{\mathbb{R}^3} |\nabla u|^2) \Delta u = k(x) \frac{|u|^2 u}{|x|} + \lambda h(x)u, & x \in \mathbb{R}^3 \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty \end{cases} \quad (1.7)$$

where  $b > 0$ ,  $h(x) > 0$  and  $k(x)$  is indefinite. In order to state our main results, we assume the following hypotheses (H):

(H<sub>h</sub>)  $h \in L^{\frac{3}{2}}(\mathbb{R}^3)$ ,  $h(x) \geq 0$  for any  $x \in \mathbb{R}^3$ ;

(H<sub>k<sub>1</sub></sub>)  $k(x) \in C(\mathbb{R}^3)$  and  $k(x)$  changes sign in  $\mathbb{R}^3$ ;

(H<sub>k<sub>2</sub></sub>)  $\lim_{|x| \rightarrow \infty} k(x) = k_\infty < 0$ ,  $k(0) = 0$ .

As far as we know, no one considered this case before. Under hypothesis (H<sub>h</sub>), there exists a sequence of eigenvalues  $\lambda_n$  of

$$-\Delta u + u = \lambda h(x)u \quad \text{in } H^1(\mathbb{R}^3)$$

with  $0 < \lambda_1 < \lambda_2 \leq \dots$  and each eigenvalue being of finite multiplicity. The associated normalized eigenfunctions are denoted by  $e_1, e_2, \dots$  with  $\|e_i\| = 1$ . Moreover,  $e_1 > 0$  in  $\mathbb{R}^3$ .

We are now ready to state our results:

**Theorem 1** *Assume that hypotheses (H) hold. Then for  $0 < \lambda < \lambda_1$ , problem (1.7) has at least one solution in  $D^{1,2}(\mathbb{R}^3)$ .*

**Theorem 2** *Assume that hypotheses (H) hold and  $\int_{\mathbb{R}^3} \frac{k(x)}{|x|} e_1^4 dx - b(\int_{\mathbb{R}^3} |\nabla e_1|^2 dx)^2 < 0$ . Then there exists  $\delta > 0$  such that problem (1.7) has at least two solutions whenever  $\lambda_1 < \lambda < \lambda_1 + \delta$ .*

**Remark 1.** Comparing with problem (1.4), we mainly consider the problem in the whole space  $\mathbb{R}^3$  with  $q = 2, p = 4$ . In this sense, our main results can be viewed as partial extensions of the results of [11].

**Remark 2.** To the best of our knowledge, for the semilinear elliptic equations with indefinite nonlinearity, a similar condition like  $\int_{\mathbb{R}^3} k(x) e_1^q dx < 0$  is needed (e.g. see [1, 7, 5]). In [12], the authors proved similar results for equation (1.5) as  $4 < p < 6$ , and the condition  $\int_{\mathbb{R}^3} \frac{k(x)}{|x|} e_1^4 dx - b(\int_{\mathbb{R}^3} |\nabla e_1|^2 dx)^2 < 0$  was not needed. However, he needed another condition:

$$(A2) : |\Omega^0| = 0 \text{ where } \Omega^0 = \{x \in \mathbb{R}^3 : k(x) = 0\}.$$

By using the same argument in this paper, it is much easier to get the same result for equation (1.5) when  $p = 4$ . In this sense, our main results can be viewed as a partial extension of the result of [12].

**Remark 3.** Comparing with problem (1.6), we mainly consider the case that  $p = 4, s = 1$ . In this sense, our main results can be views as a partial extension of the results of [17].

**Remark 4.** For system (1.1), when the nonlinearity is subcritical, as far as we know, no one consider the "zero mass" case, that is  $V(x) = 0$ .

To prove Theorems 1.1 and 1.2, we use the Nehari manifold method borrowing from Brown and Zhang [5]. In [5], the authors considered a semilinear boundary value problem on a bounded domain. Together with a concentration-compactness principle, Chabrowski and Costa [8, 10] generalized the result to unbounded region and singular nonlinearity respectively. Inspired by the papers of Brown -Zhang [5] and Chabrowski-Costa [8, 10], we extend the results to the Kirchhoff equation in  $\mathbb{R}^3$ .

## 2 Preliminaries

To go further, let us give some notions and some known results.

\*  $D^{1,2}(\mathbb{R}^3)$  is the completion of  $C_0^\infty(\mathbb{R}^3)$  with respect to the norm  $\|u\|^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx$ .

\*  $\|\cdot\|$  denotes the norm of  $D^{1,2}(\mathbb{R}^3)$ .

\*  $\rightarrow$  denotes the strong convergence.

\*  $\rightharpoonup$  denotes the weak convergence.

\*  $C, C_i$  and  $c$  denote various positive constants.

## 2.1 $C^1$ functional

For  $u \in D^{1,2}(\mathbb{R}^3)$ , weak solutions to (1.7) correspond to critical points of the energy functional

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x) u^2 dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 - \frac{1}{4} \int_{\mathbb{R}^3} \frac{k(x) |u|^4}{|x|} dx.$$

By the Caffarelli-Kohn-Nirenberg inequality [6]

$$C \left( \int_{\mathbb{R}^3} |x|^{-qb} |u|^q \right)^{\frac{2}{q}} \leq \int_{\mathbb{R}^3} |x|^{-pa} |\nabla u|^p dx.$$

where  $u \in C_0^\infty(\mathbb{R}^3)$ ,  $1 < p < N$ ,  $0 \leq a \leq b \leq a+1 \leq \frac{N}{p}$ ,  $q := \frac{NP}{N+p(b-a)-p}$ . Let  $u \in D^{1,2}(\mathbb{R}^3)$ , by approximation, it is easy to see that there exists a constant  $C$  such that

$$C \left( \int_{\mathbb{R}^3} |x|^{-1} |u|^4 \right)^{\frac{1}{2}} \leq \int_{\mathbb{R}^3} |\nabla u|^2 dx. \quad (2.1)$$

(Let  $a = 0$ ,  $b = \frac{1}{4}$ ,  $N = 3$ ,  $p = 2$ , we have  $q := p_* = 4$ .)

By (2.1), it is no difficult to show that the functional  $J$  is of class  $C^1$  (See Lemma 2.2). Moreover,

$$J'(u)v = \int_{\mathbb{R}^3} \nabla u \nabla v - \lambda h(x) u v dx + \int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \nabla u \nabla v dx - \int_{\mathbb{R}^3} \frac{k(x)}{|x|} |u|^2 u v dx$$

for any  $v \in D^{1,2}(\mathbb{R}^3)$ .

In order to use the critical point theory, we need to prove the energy functional  $J(u)$  is of a class of  $C^1$  in  $D^{1,2}(\mathbb{R}^3)$ .

**Lemma 2.1** *If  $u_n \rightarrow u$  in  $D^{1,2}(\mathbb{R}^3)$ , there exists a subsequence, still denoted by  $u_n$  and  $g \in D^{1,2}(\mathbb{R}^3)$  such that  $u_n \rightarrow u$  almost everywhere on  $\mathbb{R}^3$  and*

$$|u_n| \leq g, |u| \leq g.$$

*Proof.* Going if necessary to a subsequence, we can assume that  $u_n \rightarrow u$  a.e. on  $\mathbb{R}^3$ . There exists a subsequence, still denoted by  $u_n$  such that

$$\|u_{j+1} - u_j\| \leq 2^{-j}, \quad \forall j \geq 1.$$

Let us define

$$g(x) = |u_1(x)| + \sum_{j=1}^{\infty} |u_{j+1}(x) - u_j(x)|.$$

It is clear that  $|u_n| \leq g, |u| \leq g$  a.e. on  $\mathbb{R}^3$  and  $g \in D^{1,2}(\mathbb{R}^3)$ .

**Remark.** In order to use the Lebesgue convergence theorem and Caffarelli-Kohn-Nirenberg inequality, in the proof of Lemma 2.2 we require  $g \in D^{1,2}(\mathbb{R}^3)$ . Usually, we only require  $g \in L^p(\mathbb{R}^3)$  for  $1 \leq p < \infty$ ; see Lemma A.1 of Appendix A in [18]. In this sense, the lemma seems to be new.

**Lemma 2.2**  *$J$  is of class  $C^1$  in  $D^{1,2}(\mathbb{R}^3)$ .*

*Proof.* Let  $\varphi(x) = \int_{\mathbb{R}^3} \frac{u^4}{|x|} dx$ . We only need to prove  $\varphi(u)$  is of  $C^1$  class.

First, we prove to *Existence of the Gateaux derivative of  $\varphi$  at  $u$* .

Let  $u, v \in D^{1,2}(\mathbb{R}^3)$  and  $t \in (0, 1)$ . Since

$$(u + tv)^4 = u^4 + C_4^1 u(tv)^3 + C_4^2 u^2(tv)^2 + C_4^3 u^3(tv)^1 + (tv)^4.$$

We have

$$\begin{aligned} \frac{|((u+tv)^4 - u^4)|}{|x|t} &\leq \frac{4t^2|u||v|^3 + 6t|u|^2|v|^2 + 4|u|^3|v| + t^3|v|^4}{|x|} \\ &\leq 6 \frac{|u||v|^3 + |u|^2|v|^2 + |u|^3|v| + |v|^4}{|x|}. \end{aligned}$$

By Hölder inequality and Caffarelli-Kohn-Nirenberg inequality (2.1), we have

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|u||v|^3}{|x|} dx &= \int_{\mathbb{R}^3} \frac{|u|}{|x|^{\frac{1}{4}}} \frac{|v|^3}{|x|^{\frac{3}{4}}} dx \\ &\leq \left( \int_{\mathbb{R}^3} \left( \frac{|v|^3}{|x|^{\frac{3}{4}}} \right)^{\frac{4}{3}} dx \right)^{\frac{3}{4}} \left( \int_{\mathbb{R}^3} \left( \frac{|u|}{|x|^{\frac{1}{4}}} \right)^4 dx \right)^{\frac{1}{4}} \\ &\leq C \left( \int_{\mathbb{R}^3} |\nabla v|^2 dx \right)^{\frac{3}{2}} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{\frac{1}{2}} \\ &\leq C_1. \end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^3} \frac{|u|^2|v|^3}{|x|} dx \leq C_2, \int_{\mathbb{R}^3} \frac{|u|^3|v|}{|x|} dx \leq C_3, \int_{\mathbb{R}^3} \frac{|v|^4}{|x|} dx \leq C_4.$$

Therefore, we have

$$\frac{|((u+tv)^4 - u^4)|}{|x|t} \leq \eta(v) \in L^1(\mathbb{R}^3)$$

where  $\eta(v) = 6 \frac{|u||v|^3 + |u|^2|v|^2 + |u|^3|v| + |v|^4}{|x|}$ . It follows then from the Lebesgue convergence theorem that

$$\langle \varphi'(u), v \rangle = 4 \int_{\mathbb{R}^3} \frac{|u|^2 uv}{|x|} dx.$$

Next, we prove the *Gateaux derivation is continuous*.

Assume that  $u_n \rightarrow u$  in  $D^{1,2}(\mathbb{R}^3)$ . By Hölder inequality and Caffarelli-Kohn-Nirenberg inequality (2.1), we have

$$\begin{aligned} |\langle \varphi'(u_n) - \varphi'(u), v \rangle| &= \int_{\mathbb{R}^3} \frac{(|u_n|^2 u_n - |u|^2 u)v}{|x|} dx \\ &= \int_{\mathbb{R}^3} \frac{|v|}{|x|^{\frac{1}{4}}} \frac{(|u_n|^2 u_n - |u|^2 u)v}{|x|^{\frac{3}{4}}} dx \\ &\leq C \left( \int_{\mathbb{R}^3} \left( \frac{(|u_n|^2 u_n - |u|^2 u)v}{|x|^{\frac{3}{4}}} \right)^{\frac{4}{3}} dx \right)^{\frac{3}{4}} \|v\|. \end{aligned}$$

Lemma 2.1 implies that there exists  $g \in D^{1,2}(\mathbb{R}^3)$  such that  $|u_n| \leq g, |u| \leq g$ . By Caffarelli-Kohn-Nirenberg inequality (2.1), we have

$$\left( \frac{(|u_n|^2 u_n - |u|^2 u)}{|x|^{\frac{3}{4}}} \right)^{\frac{4}{3}} \leq \frac{2g^4}{|x|} \in L^1(\mathbb{R}^3).$$

According to Lebesgue convergence theorem, we get

$$\|\varphi'(u_n) - \varphi'(u)\| \leq C \left( \int_{\mathbb{R}^3} \left( \frac{(|u_n|^2 u_n - |u|^2 u)}{|x|^{\frac{3}{4}}} \right)^{\frac{4}{3}} dx \right)^{\frac{3}{4}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

## 2.2 Nehari manifold

For  $u \in D^{1,2}(\mathbb{R}^3)$ , weak solutions to (1.7) correspond to critical points of the energy functional

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x) u^2 dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 - \frac{1}{4} \int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx.$$

Since the functional  $J$  is not bounded from below on  $D^{1,2}(\mathbb{R}^3)$ , a good candidate for an appropriate subset to study  $J$  is the so-called Nehari manifold

$$\begin{aligned} S &= \{u \in D^{1,2}(\mathbb{R}^3) \mid J'(u)u = 0\} \\ &= \{u \in D^{1,2}(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x) u^2 dx = \int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx - \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2\}. \end{aligned}$$

It is useful to understand  $S$  in term of the stationary points of the fibering mappings, i.e.

$$\varphi_u(t) = J(tu) = \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x) u^2 dx - \frac{t^4}{4} \int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx + \frac{bt^4}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2.$$

We now follow some ideas from the paper [5].

**Lemma 2.3** *Let  $u \in D^{1,2}(\mathbb{R}^3) - \{0\}$  and  $t > 0$ . Then  $tu \in S$  if and only if  $\varphi'_u(t) = 0$ .*

Thus the points in  $S$  correspond to the stationary points of the fiber map  $\varphi_u(t)$  and so it is natural to divide  $S$  into three parts  $S^+$ ,  $S^-$  and  $S^0$  corresponding to local minima, local maxima and points of inflexion of the fibering maps. We have

$$\varphi''_u(t) = \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x) u^2 dx - 3t^2 \left( \int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 \right)$$

and

$$\varphi''_u(1) = \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x) u^2 dx - 3 \left( \int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 \right) dx.$$

Hence if we define

$$\begin{aligned} S^+ &= \{u \in S : \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x) u^2 dx - 3 \left( \int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 \right) dx > 0\}, \\ S^- &= \{u \in S : \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x) u^2 dx - 3 \left( \int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 \right) dx < 0\}, \\ S^0 &= \{u \in S : \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x) u^2 dx - 3 \left( \int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 \right) dx = 0\}, \end{aligned}$$

we have

**Lemma 2.4** *Let  $u \in S$ . Then*

$$\begin{aligned} S^+ &= \{u \in S : \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x) u^2 dx < 0\} \\ &= \{u \in S : \int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 < 0\}, \\ S^- &= \{u \in S : \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x) u^2 dx > 0\} \end{aligned}$$

$$= \{u \in S : \int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx - b(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 > 0\},$$

$$\begin{aligned} S^0 &= \{u \in S : \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x)u^2 dx = 0\} \\ &= \{u \in S : \int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx - b(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 = 0\}. \end{aligned}$$

Since

$$\varphi'_u(t) = t \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x)u^2 dx - t^3 \left( \int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx - b(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 \right),$$

$\varphi_u$  has exactly one turning point at  $t(u) = \left\{ \frac{\int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x)u^2 dx}{\int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx - b(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2} \right\}^{\frac{1}{2}}$  if and only if  $\int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x)u^2 dx$  and  $\int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx - b(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2$  have the same sign.

As in [5], we let

$$\begin{aligned} L^+ &= \{u \in H^1(\mathbb{R}^3) : \|u\| = 1, \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x)u^2 dx > 0\}, \\ L^- &= \{u \in H^1(\mathbb{R}^3) : \|u\| = 1, \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x)u^2 dx < 0\}, \\ L^0 &= \{u \in H^1(\mathbb{R}^3) : \|u\| = 1, \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x)u^2 dx = 0\}, \end{aligned}$$

and

$$\begin{aligned} B^+ &= \{u \in H^1(\mathbb{R}^3) : \|u\| = 1, \int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx - b(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 > 0\}, \\ B^- &= \{u \in H^1(\mathbb{R}^3) : \|u\| = 1, \int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx - b(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 < 0\}, \\ B^0 &= \{u \in H^1(\mathbb{R}^3) : \|u\| = 1, \int_{\mathbb{R}^3} \frac{k(x)|u|^4}{|x|} dx - b(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 = 0\}. \end{aligned}$$

**Lemma 2.5** (i). A multiple of  $u$  lies in  $S^-$  if and only if  $\frac{u}{\|u\|}$  lies in  $L^+ \cap B^+$ .  
(ii). A multiple of  $u$  lies in  $S^+$  if and only if  $\frac{u}{\|u\|}$  lies in  $L^- \cap B^-$ .  
(iii). For  $u \in L^+ \cap B^-$  or  $u \in L^- \cap B^+$ , no multiple of  $u$  lies in  $S$ .

**Theorem 3** Suppose that  $u_0$  is a local minimizer for  $J$  on  $S$  and  $u_0 \notin S^0$ , then  $J'(u_0) = 0$ .

## 2.3 concentration-compactness principle

In order to overcome the loss of compactness we make use of a simple version of concentration-compactness principle which is considered in [9, Proposition 1.3], see also [10, Concentration-Compactness Principle].

**Theorem 4** Let  $(u_m)$  be a sequence in  $D^{1,2}(\mathbb{R}^3)$  such that

$$\begin{aligned} u_m(x) &\rightarrow u(x) \quad \text{a.e. in } \mathbb{R}^3, \\ u_m(x) &\rightharpoonup u(x) \quad \text{in } D^{1,2}(\mathbb{R}^3), \\ |\nabla(u_m - u)|^2 &\rightharpoonup \mu \quad \text{in } M(\mathbb{R}^3), \\ |x|^{-\frac{1}{4}}|u_m - u|^4 &\rightharpoonup \nu \quad \text{in } M(\mathbb{R}^3), \end{aligned}$$

where  $M(\mathbb{R}^3)$  denotes the space of bounded measures in  $\mathbb{R}^3$ . Define the quantities

$$\alpha_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} \frac{|u_n|^4}{|x|} dx$$

$$\beta_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |\nabla u_n|^2 dx.$$

$$\nu_0 = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| \leq \frac{1}{R}} \frac{|u_n|^4}{|x|} dx$$

$$\mu_0 = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| \leq \frac{1}{R}} |\nabla u_n|^2 dx.$$

Then we have

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = \int_{\mathbb{R}^3} |\nabla u|^2 dx + \beta_\infty + \mu_0. \quad (2.2)$$

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \frac{|u_n|^4}{|x|} dx = \int_{\mathbb{R}^3} \frac{|u|^4}{|x|} dx + \alpha_\infty + \nu_0. \quad (2.3)$$

### 3 The case when $0 < \lambda < \lambda_1$

Suppose that  $0 < \lambda < \lambda_1$ . It is easy to see that there exists  $\theta > 0$  such that

$$\int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x) u^2 dx \geq \theta \|u\|^2. \quad (3.1)$$

Thus  $S^+$  is empty and  $S^0 = \{0\}$ .

To prove the theorem, we need the following lemma.

**Lemma 3.1** *Suppose  $0 < \lambda < \lambda_1$ . Then*

(i).  $\inf_{u \in S^-} J(u) > 0$ ;

(ii). *There exists  $u \in S^- \setminus \{0\}$ , such that  $J(u) = \inf_{v \in S^-} J(v)$ .*

*Proof.* (i). By Lemma 2.4, we have

$$J(u) = \frac{1}{4} \int_{\mathbb{R}^3} k(x) |u|^4 - l(x) \phi_u(x) u^2 dx > 0 \quad \text{when } u \in S^-.$$

So  $J$  is bounded below by 0 on  $S^-$ . We show that  $\inf_{u \in S^-} J(u) > 0$ . Suppose  $u \in S^-$ . Then  $v = \frac{u}{\|u\|} \in L^+ \cap B^+$  and  $u = t(v)v$  where  $t(v) = [\frac{\int_{\mathbb{R}^3} |\nabla v|^2 - \lambda h(x) v^2 dx}{\int_{\mathbb{R}^3} k(x) \frac{|v|^4}{|x|} dx - b(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2}]^{\frac{1}{2}}$ . In addition,

$$\begin{aligned} J(u) = J(t(v)v) &= \frac{1}{4} (t(v))^2 \int_{\mathbb{R}^3} |\nabla v|^2 + |v|^2 - \lambda h(x) v^2 dx \\ &= \frac{1}{4} \frac{(\int_{\mathbb{R}^3} |\nabla v|^2 - \lambda h(x) v^2 dx)^2}{\int_{\mathbb{R}^3} k(x) \frac{|v|^4}{|x|} dx - b(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2} \\ &\geq \frac{\theta^2}{4 \int_{\mathbb{R}^3} k(x) \frac{|v|^4}{|x|} dx - b(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2} \quad (\text{by (3.1)}) \\ &\geq \frac{\theta^2}{4 \int_{\mathbb{R}^3} k(x) \frac{|v|^4}{|x|} dx}. \end{aligned} \quad (a)$$



We now focus on the term  $\int_{\mathbb{R}^3} k(x) \frac{|v|^4}{|x|} dx$ . By Caffarelli-Kohn-Nirenberg inequality (2.1), we have

$$\begin{aligned} \int_{\mathbb{R}^3} k(x) \frac{|v|^4}{|x|} dx &\leq C|k|_{L^\infty} \int_{\mathbb{R}^3} \frac{|v|^4}{|x|} dx \\ &\leq C|k|_{L^\infty} \|v\|^2 \\ &= C|k|_{L^\infty}. \end{aligned} \tag{b}$$

Combining (a) with (b), we have

$$J(u) \geq \frac{\theta^2}{C|k|_{L^\infty}} > 0.$$

Hence

$$\inf_{u \in S^-} J(u) > 0.$$

(ii). We show that there exists a minimizer on  $S^-$ . Let  $\{u_n\} \subset S^-$  be a minimizing sequence, i.e.,  $\lim_{n \rightarrow \infty} J(u_n) = \inf_{u \in S^-} J(u)$ . By (3.1), we have

$$J(u_n) = \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u_n|^2 - \lambda h(x) u_n^2 dx \geq \frac{1}{4} \theta \|u_n\|^2.$$

So  $\{u_n\}$  is bounded in  $D^{1,2}(\mathbb{R}^3)$ . Passing to a subsequence if necessary, we obtain that  $\{u_n\} \rightharpoonup u$  in  $D^{1,2}(\mathbb{R}^3)$ . First, we claim that  $u \neq 0$ . Since  $\{u_n\} \subset S$ , we have

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 - \lambda h(x) u_n^2 dx = \int_{\mathbb{R}^3} k(x) \frac{|u_n|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2.$$

Using (2.2) and (2.3), we deduce

$$\int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x) |u|^2 dx + \beta_\infty + \mu_0 \leq \int_{\mathbb{R}^3} k(x) \frac{|u|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 + k(0)\nu_0 + k(\infty)\nu_\infty. \tag{3.2}$$

Suppose  $u = 0$ . By (3.2), we have

$$\beta_\infty = \mu_0 = 0.$$

Then  $u_n \rightarrow 0$  in  $D^{1,2}(\mathbb{R}^3)$ , a contradiction to  $\inf_{u \in S^-} J(u) > 0$ .

We now claim that  $\beta_\infty = \mu_0 = 0$ . Otherwise, we deduce from (3.2) that

$$0 < \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x) |u|^2 dx < \int_{\mathbb{R}^3} k(x) \frac{|u|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2.$$

There exists  $0 < s < 1$  such that

$$\int_{\mathbb{R}^3} |\nabla su|^2 - \lambda h(x) |su|^2 dx = \int_{\mathbb{R}^3} \frac{k(x)}{|x|} (su)^4 dx - b \left( \int_{\mathbb{R}^3} |\nabla su|^2 dx \right)^2.$$

This implies that  $su$  belongs to  $S^-$ . On other hand, since

$$\begin{aligned} 0 < \int_{\mathbb{R}^3} |\nabla u|^2 - \lambda h(x) |u|^2 dx &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 - \lambda h(x) |u_n|^2 dx \\ &= 4 \inf_{w \in S^-} J(w) \\ &\leq \int_{\mathbb{R}^3} |\nabla su|^2 - \lambda h(x) |su|^2 dx, \end{aligned}$$

we have  $s \geq 1$ , a contradiction. Consequently, we have  $u_n \rightarrow u$  in  $D^{1,2}(\mathbb{R}^3)$ . With the help of the preceding lemmas we can now prove Theorem 1.1.

**Proof of Theorem 1.1.** The theorem follows immediately from Lemma 3.1 and Theorem 2.1.

## 4 The case when $\lambda > \lambda_1$

We assume that  $\int_{\mathbb{R}^3} k(x)e_1^4 dx - b(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 < 0$ . Then  $e_1 \in L^- \cap B^-$  and  $t(e_1)e_1 \in S^+$ . The following lemma plays an important role for establishing the existence of minimizers.

**Lemma 4.1** *Suppose that  $\int_{\mathbb{R}^3} \frac{k(x)}{|x|} |e_1|^4 dx - b(\int_{\mathbb{R}^3} |\nabla e_1|^2 dx)^2 < 0$ . Then there exists  $\sigma > 0$  such that  $\overline{L^-} \cap \overline{B^+} = \emptyset$  whenever  $\lambda_1 < \lambda < \lambda_1 + \sigma$ .*

*Proof.* By contradiction, then there exist sequences  $\{\lambda_n\}$  and  $\{u_n\}$  such that  $\|u_n\| = 1$ ,  $\lambda_n \rightarrow \lambda_1^+$  and

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla u_n|^2 - \lambda_n h(x) u_n^2 dx &\leq 0, \\ \int_{\mathbb{R}^3} \frac{k(x)}{|x|} |u_n|^4 dx - b(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 &\geq 0. \end{aligned}$$

Since  $u_n$  is bounded, we may assume that  $u_n \rightharpoonup u$ . We show that  $u_n \rightarrow u$  in  $D^{1,2}(\mathbb{R}^3)$ . Supposing otherwise, then we have  $\|u\| < \liminf_{n \rightarrow \infty} \|u_n\|$  and

$$\int_{\mathbb{R}^3} |\nabla u|^2 - \lambda_1 h(x) |u|^2 < \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 - \lambda_n h(x) |u_n|^2 \leq 0.$$

This is a contradiction to  $\lambda_1$ . Hence  $u_n \rightarrow u$  in  $D^{1,2}(\mathbb{R}^3)$ ,  $\|u\| = 1$  and  $u = \pm e_1$ . To get a contradiction, the cases  $u = e_1$  and  $u = -e_1$  are entirely similar, so that we only consider  $u = e_1$ . On the other hand, since functional  $b(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2$  is continuous on  $D^{1,2}(\mathbb{R}^3)$ , we have

$$\lim_{n \rightarrow \infty} b(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx)^2 = b(\int_{\mathbb{R}^3} |\nabla e_1|^2 dx)^2.$$

By Caffarelli-Kohn-Nirenberg inequality (2.1), we have

$$\int_{\mathbb{R}^3} \frac{k(x)}{|x|} |u_n|^4 dx \rightarrow \int_{\mathbb{R}^3} \frac{k(x)}{|x|} |e_1|^4 dx.$$

Hence,  $\int_{\mathbb{R}^3} \frac{k(x)}{|x|} |e_1|^4 dx - b(\int_{\mathbb{R}^3} |\nabla e_1|^2 dx)^2 \geq 0$ , This is a contradiction to our assumption. If  $\overline{L^-} \cap \overline{B^+} = \emptyset$  is satisfied, we can get more information on  $S$ .

**Lemma 4.2** *Suppose that  $\overline{L^-} \cap \overline{B^+} = \emptyset$ . Then*

- (i)  $S^0 = \{0\}$ .
- (ii)  $0 \notin \overline{S^-}$  and  $S^-$  is closed.
- (iii)  $S^-$  and  $S^+$  are separated. That is  $\overline{S^-} \cap \overline{S^+} = \emptyset$ .
- (iv)  $S^+$  is bounded.

*Proof.* (i). Suppose  $u \in S^0 \setminus \{0\}$ . Then  $\frac{u}{\|u\|} \in L^0 \cap B^0 \subset \overline{L^-} \cap \overline{B^+} = \emptyset$  which is impossible. Hence  $S^0 = \{0\}$ .

(ii). Supposing otherwise, then there exists  $\{u_n\} \in S^-$  such that  $u_n \rightarrow 0$  in  $D^{1,2}(\mathbb{R}^3)$ . Hence

$$0 < \int_{\mathbb{R}^3} |\nabla u_n|^2 - \lambda h(x) u_n^2 dx = \int_{\mathbb{R}^3} k(x) \frac{|u_n|^4}{|x|} dx - b(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx)^2 \rightarrow 0. \quad (4.1)$$

Let  $v_n = \frac{u_n}{\|u_n\|}$ . We observe that

$$0 < \int_{\mathbb{R}^3} |\nabla v_n|^2 - \lambda h(x) v_n^2 dx = \|u_n\|^2 (\int_{\mathbb{R}^3} k(x) \frac{|v_n|^4}{|x|} dx - b(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx)^2). \quad (4.2)$$

We may assume that  $v_n \rightharpoonup v_0$  in  $D^{1,2}(\mathbb{R}^3)$ . To obtain a contradiction, we divide our proof into three steps:

- (a)  $v_0 \neq 0$ .
- (b)  $\frac{v_0}{\|v_0\|} \in \overline{L^-}$ .
- (c)  $\frac{v_0}{\|v_0\|} \in \overline{B^+}$ .

We begin to prove the assertions (a), (b) and (c).

(a). By Caffarelli-Kohn-Nirenberg inequality (2.1), we obtain

$$\int_{\mathbb{R}^3} k(x) \frac{|v_n|^4}{|x|} dx \leq C \|v_n\|^4 = C. \quad (4.3)$$

Then we have

$$0 < \int_{\mathbb{R}^3} k(x) \frac{|v_n|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla v_n|^2 dx \right)^2 \leq C. \quad (4.4)$$

Since  $u_n \in S^-$  and  $\|u_n\| \rightarrow 0$ , by (4.2) and (4.4), we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla v_n|^2 - \lambda h(x) v_n^2 dx = 0, \quad (4.5)$$

that is

$$1 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \lambda h(x) v_n^2 dx = \int_{\mathbb{R}^3} \lambda h(x) v_0^2 dx.$$

So  $v_0 \neq 0$ .

(b). By (4.5), we have

$$\int_{\mathbb{R}^3} |\nabla v_0|^2 - \lambda h(x) v_0^2 dx \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla v_n|^2 - \lambda h(x) v_n^2 dx = 0.$$

So  $\frac{v_0}{\|v_0\|} \in \overline{L^-}$ .

(c). According to (4.4), it follows that

$$\begin{aligned} 0 &\leq \limsup_{n \rightarrow \infty} \left( \int_{\mathbb{R}^3} k(x) \frac{|v_n|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla v_n|^2 dx \right)^2 \right) \leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} k(x) \frac{|v_n|^4}{|x|} dx - \liminf_{n \rightarrow \infty} b \left( \int_{\mathbb{R}^3} |\nabla v_n|^2 dx \right)^2 \\ &\leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} k(x) \frac{|v_n|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla v_0|^2 dx \right)^2. \end{aligned}$$

Therefore, according to the definitions of  $\alpha_\infty, k_\infty, \nu_0$ , we have

$$\int_{\mathbb{R}^3} k(x) \frac{|v_0|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla v_0|^2 dx \right)^2 \geq -k_\infty \alpha_\infty - k(0) \nu_0 \geq 0.$$

Hence,  $0 \notin \overline{S^-}$ . Now we are ready to prove that  $S^-$  is closed. Let  $u \in \overline{S^-}$ , then there exist  $\{u_n\} \subseteq S^-$  such that  $u_n \rightarrow u$  in  $H^1(\mathbb{R}^3)$ . Since  $S$  is closed,  $S^+$  is open,  $0 \notin \overline{S^-}$  and  $S = S^- \cup S^+ \cup \{0\}$ , we have  $u \in S^-$ . We conclude that  $S^-$  is closed.

(iii). By (i) and (ii), we have

$$\overline{S^-} \cap \overline{S^+} \subseteq S^- \cap (S^+ \cup \{0\}) = (S^- \cap S^+) \cup (S^- \cap \{0\}) = \emptyset.$$

(iv). Suppose that  $S^+$  is unbounded. Then there exists a sequence  $\{u_n\} \subset S^+$  such that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 - \lambda h(x) u_n^2 dx = \int_{\mathbb{R}^3} k(x) \frac{|u_n|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 \leq 0$$

and  $\|u_n\| \rightarrow \infty$ . Let  $v_n = \frac{u_n}{\|u_n\|}$ . We may assume that  $v_n \rightharpoonup v_0$  in  $D^{1,2}(\mathbb{R}^3)$ . It is clear that

$$\int_{\mathbb{R}^3} |\nabla v_n|^2 - \lambda h(x) v_n^2 dx = \|u_n\|^2 \left( \int_{\mathbb{R}^3} k(x) \frac{|v_n|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla v_n|^2 dx \right)^2 \right) \quad (4.6)$$

and also

$$\int_{\mathbb{R}^3} |\nabla v_0|^2 - \lambda h(x) v_0^2 dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla v_n|^2 - \lambda h(x) v_n^2 dx \leq 0. \quad (4.7)$$

We shall adopt the same procedure as (a),(b),(c) in (ii). Suppose  $v_0 = 0$ . We claim that  $v_n \rightarrow 0$  in  $D^{1,2}(\mathbb{R}^3)$ . Indeed, using (4.7), if  $v_n \not\rightarrow 0$ , we have

$$0 = \int_{\mathbb{R}^3} |\nabla v_0|^2 - \lambda h(x) v_0^2 dx < \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla v_n|^2 - \lambda h(x) v_n^2 dx \leq 0.$$

Therefore,  $v_n \rightarrow 0$  in  $H^1(\mathbb{R}^3)$ , which a contradiction to  $\|v_n\| = 1$ . So  $\|v_0\| \neq 0$ .

By (4.7) and  $v_0 \neq 0$ , we have

$$\frac{v_0}{\|v_0\|} \in \overline{L^-}.$$

On the other hand, according to (4.6), it follows that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} k(x) \frac{|v_n|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla v_n|^2 dx \right)^2 = 0.$$

According to the definitions of  $\alpha_\infty$  and  $k_\infty$ , then we have

$$\begin{aligned} \int_{\mathbb{R}^3} k(x) \frac{|v_0|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla v_0|^2 dx \right)^2 &\geq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} k(x) \frac{|v_n|^4}{|x|} dx - \liminf_{n \rightarrow \infty} b \left( \int_{\mathbb{R}^3} |\nabla v_n|^2 dx \right)^2 - (\alpha_\infty k_\infty + k(0) \nu_0) \\ &\geq \lim_{n \rightarrow \infty} \left( \int_{\mathbb{R}^3} k(x) \frac{|v_n|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla v_n|^2 dx \right)^2 \right) - (\alpha_\infty k_\infty + k(0) \nu_0) \\ &= -(\alpha_\infty k_\infty + k(0) \nu_0) \geq 0. \end{aligned}$$

This means  $\frac{v_0}{\|v_0\|} \in \overline{B^+}$ .

Summarizing what have proved, (a).  $v_0 \neq 0$ , (b).  $\frac{v_0}{\|v_0\|} \in \overline{L^-}$ , (c).  $\frac{v_0}{\|v_0\|} \in \overline{B^+}$ , it is a contradiction to  $\overline{L^-} \cap \overline{B^+} = \emptyset$ . Then  $S^+$  is bounded.

**Lemma 4.3** Suppose that  $\overline{L^-} \cap \overline{B^+} = \emptyset$ . Then

- (i). Every minimizing sequence of  $J(u)$  on  $S^-$  is bounded.
- (ii).  $\inf_{u \in S^-} J(u) > 0$ .
- (iii). There exists a minimizer of  $J(u)$  on  $S^-$ .

*Proof.* (i). Let  $\{u_n\} \in S^-$  be a minimizing sequence for the functional  $J(u)$ . Then

$$J(u_n) = \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u_n|^2 - \lambda h(x) u_n^2 dx = \frac{1}{4} \left( \int_{\mathbb{R}^3} k(x) \frac{|u_n|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 \right) \rightarrow c,$$

where  $c > 0$ . Similar to the one used in the proof of (iv) of Lemma 4.2, it is easy to prove that  $\{u_n\}$  is bounded in  $D^{1,2}(\mathbb{R}^3)$ .

(ii). It is clear that  $J(u) \geq 0$  on  $S^-$ . Suppose  $\inf_{w \in S^-} J(w) = 0$ . Let  $\{u_n\}$  be a minimizing sequence. According to (i), it follows that  $\{u_n\}$  is bounded and we may assume  $u_n \rightharpoonup u_0$ . Clearly, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 - \lambda h(x) u_n^2 dx = 0 \quad (4.8)$$

and

$$\lim_{n \rightarrow \infty} \left( \int_{\mathbb{R}^3} k(x) \frac{|u_n|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 \right) = 0. \quad (4.9)$$

Then it is easy to prove that  $\frac{u_0}{\|u_0\|} \in \overline{B^+} \cap \overline{L^-}$ . Indeed, using the same argument in Lemma 4.2, by (4.8), it is easy to show that  $\frac{u_0}{\|u_0\|} \in \cap \overline{L^-}$  and by (4.9),  $\frac{u_0}{\|u_0\|} \in \overline{B^+}$  follows.

So  $\inf_{u \in S^-} J(u) > 0$ .

(iii). Let  $\{u_n\}$  be a minimizing sequence. According to (i), it follows that  $\{u_n\}$  is bounded and we may assume  $u_n \rightharpoonup u_0$  in  $H^1(\mathbb{R}^3)$ . Suppose  $u_n \not\rightharpoonup u_0$  in  $D^{1,2}(\mathbb{R}^3)$ . We get

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla u_0|^2 - \lambda h(x) u_0^2 dx &< \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 - \lambda h(x) u_n^2 dx \\ &= \lim_{n \rightarrow \infty} \left( \int_{\mathbb{R}^3} k(x) \frac{|u_n|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 \right) \\ &\leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} k(x) \frac{|u_n|^4}{|x|} dx - \liminf_{n \rightarrow \infty} b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 \\ &\leq \int_{\mathbb{R}^3} k(x) \frac{|u_0|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u_0|^2 dx \right)^2 + k_\infty \alpha_\infty + k(0) \nu_0 \\ &\leq \int_{\mathbb{R}^3} k(x) \frac{|u_0|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u_0|^2 dx \right)^2 \end{aligned}$$

So there exists a  $0 < t < 1$  such that  $tu_0 \in S^-$ . Similar to the proof of (ii) in Lemma 3.1, we get a contradiction.

Hence  $u_n \rightarrow u_0 \neq 0$  in  $H^1(\mathbb{R}^3)$ . Since  $S^-$  is closed, then we have  $u_0 \in S^-$  and  $J(u_0) = \inf_{w \in S^-} J(w)$ .

We are going to the investigation on  $S^+$ .

**Lemma 4.4** *Suppose that  $\overline{L^-} \cap \overline{B^+} = \emptyset$ . Then there exists  $0 \neq v \in S^+$  such that  $J(v) = \inf_{u \in S^+} J(u)$ .*

*Proof.* Due to  $\overline{L^-} \cap \overline{B^+} = \emptyset$ ,  $L^- \cap B^+$  as well as  $S^+$  must be nonempty. By (iv) of Lemma 4.2, there exists  $M > 0$  such that  $\|u\| \leq M$  for all the  $u \in S^+$ . Using Caffarelli-Kohn-Nirenberg inequality (2.1), it is easy to see that  $J(u)$  is bounded from below on  $S^+$  and  $\inf_{u \in S^+} J(u) < 0$ . Let  $\{u_n\} \in S^+$  be a minimizing sequence for the functional  $J(u)$ . Then

$$J(u_n) = \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u_n|^2 - \lambda h(x) u_n^2 dx = \frac{1}{4} \left( \int_{\mathbb{R}^3} k(x) \frac{|u_n|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 \right) \rightarrow c,$$

where  $c < 0$ . We may assume  $u_n \rightharpoonup u_0$  in  $D^{1,2}(\mathbb{R}^3)$ . Obviously,

$$\int_{\mathbb{R}^3} |\nabla u_0|^2 - \lambda h(x) u_0^2 dx \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 - \lambda h(x) u_n^2 dx = c < 0.$$

So  $u_0 \neq 0$  and  $\frac{u_0}{\|u_0\|} \in L^-$ . Since  $\overline{L^-} \cap \overline{B^+} = \emptyset$ , then we have  $\frac{u_0}{\|u_0\|} \in B^-$ .

Hence  $t(u_0)u_0 \in S^+$ , where  $t(u_0) = \left[ \frac{\int_{\mathbb{R}^3} |\nabla u_0|^2 - \lambda h(x) u_0^2 dx}{\int_{\mathbb{R}^3} k(x) \frac{|u_0|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u_0|^2 dx \right)^2} \right]^{\frac{1}{2}}$ . Suppose that  $u_n \not\rightharpoonup u_0$ . By an argument similar to the one used in the proof of (iii) in Lemma 4.3, we have

$$\int_{\mathbb{R}^3} |\nabla u_0|^2 - \lambda h(x) u_0^2 dx < \int_{\mathbb{R}^3} k(x) \frac{|u_0|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u_0|^2 dx \right)^2.$$

Together with  $\int_{\mathbb{R}^3} k(x) \frac{|u_0|^4}{|x|} dx - b \left( \int_{\mathbb{R}^3} |\nabla u_0|^2 dx \right)^2 < 0$ , (since  $\frac{u_0}{\|u_0\|} \in B^-$ ). It implies that  $t(u_0) > 1$ . This is contrary with

$$J(t(u_0)u_0) < J(u_0) \leq \lim_{n \rightarrow \infty} J(u_n) = \inf_{w \in S^+} J(w).$$

Hence

$$u_n \rightarrow u_0$$

in  $D^{1,2}(\mathbb{R}^3)$ . Since the functional  $\Xi(u) = \int_{\mathbb{R}^3} k(x) \frac{|u_n|^4}{|x|} dx - b(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx)^2$  is continuous. Therefore, we have  $u_0 \in S^+$  and

$$J(u_0) = \inf_{u \in S^+} J(u).$$

We now turn to the proof of Theorem 1.2.

**Proof of Theorem 1.2** According to Lemma 4.1 and the assumptions of Theorem 1.2, it follows that  $\overline{L^-} \cap \overline{B^+} = \emptyset$ . We are ready to invoke the conclusions of Lemma 4.3 and Theorem 2.1. So, there exists  $u_1 \in S^-$  which is a critical point of  $J(u)$ . Clearly,  $J(u_1) > 0$ . Employing Lemma 4.4 and Theorem 2.1, there exists  $u_2 \in S^+$  which is a critical point of  $J(u)$ . Clearly,  $J(u_2) < 0$ . We have thus proved Theorem 1.2.

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